RESTRICTIONS AND GENERALIZED INVERSES IN LINEAR MODELS

by

S. R. Searle

Biometrics Unit, Cornell University, Ithaca, New York

Abstract

Given any generalized inverse $(\underset{\sim}{X}'\underset{\sim}{X})^-$ appropriate to normal equations $\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{b}^o = \underset{\sim}{X}'\underset{\sim}{y}$ for the linear model $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e}$, a procedure is given for obtaining from it a generalized inverse appropriate to a restricted model having restrictions $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{0}$ for $\underset{\sim}{P}'\underset{\sim}{b}$ non-estimable.

Introduction

Mazumdar, Li and Bryce (1980), for a linear model

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e} \tag{1}$$

with restrictions

$$\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{0}\ , \tag{2}$$

consider the problem of deriving a generalized inverse of $\underset{\sim}{X}'\underset{\sim}{X}$, call it $\underset{\sim}{G}_r$, such that a solution to the least squares equations for the restricted model (1) <u>and</u> (2) is $\underset{\sim}{b}_r^o = \underset{\sim}{G}_r\underset{\sim}{X}'\underset{\sim}{y}$. They achieve this by using full rank sub-matrices of $\underset{\sim}{P}'$ and $\underset{\sim}{X}'\underset{\sim}{X}$. This note suggests an alternative procedure that is based solely on generalized inverses and which requires no partitioning of matrices.

Paper No. BU-798-M in the Biometrics Unit.

The Method

Let G be any generalized inverse of $X'X$, i.e., $X'XGX'X = X'X$, and let

$$b^o = GX'y \tag{3}$$

be the corresponding solution of the normal equations $X'Xb^o = X'y$ for the unrestricted model (1). Then, using the definition

$$H = GX'X\ , \tag{4}$$

Searle (1971, Section 5.7d) shows that – with slightly amended notation – a solution of the normal equations for the restricted model (1) and (2) is

$$b_r^o = b^o + (I - H)w \tag{5}$$

for w satisfying

$$P'(I - H)w = -P'b^o\ . \tag{6}$$

We assume P' is such that $P'b$ is non-estimable and of full row rank. Hence its rows are linearly independent of each other and of rows of X, and they do not exceed $p - r_X$ in number, where X has p columns and rank $r_X < p$. Furthermore, $(I - H)$ has rank $p - r_X$. Therefore (6) has a solution for w:

$$w = -[P'(I - H)]^- P'b^o \tag{7}$$

and so in (5)

$$b_r^o = b^o - (I - H)[P'(I - H)]^- P'b^o = G_r X'y \tag{8}$$

for

$$G_r = G - (I - H)[P'(I - H)]^- P'G\ . \tag{9}$$

If a symmetric, reflexive generalized inverse is wanted in place of G_r, i.e., a G such that $G = G' = GX'XG$, then it is obtainable from G_r as

$$G_r^{\#} = G_r X'XG_r'\ . \tag{10}$$

Since the definition of $\underset{\sim}{H}$ in (4) gives $\underset{\sim}{X}'\underset{\sim}{X}(\underset{\sim}{I} - \underset{\sim}{H}) = \underset{\sim}{0}$, it is easily seen that $\underset{\sim}{G}_r$ is a generalized inverse of $\underset{\sim}{X}'\underset{\sim}{X}$; and by (8) it gives $\underset{\sim}{b}_r^o$ as $\underset{\sim}{G}_r\underset{\sim}{X}'\underset{\sim}{y}$. Thus (9) is the procedure for establishing a generalized inverse of $\underset{\sim}{X}'\underset{\sim}{X}$ that converts any solution to the normal equations into one that corresponds to restrictions $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{0}$; and (8) is that solution. No partitioning of $\underset{\sim}{P}'$ and $\underset{\sim}{X}'\underset{\sim}{X}$ is required; (9) relies solely on $\underset{\sim}{P}'$ and any $\underset{\sim}{G}$.

Example: Consider fitting $y_{ij} = \mu + \alpha_i + e_{ij}$ to the following 6 observations in 3 classes of a 1-way classification

Data		
101	84	32
105	88	
94		
300	172	32

The normal equations are

$$\begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & \cdot & \cdot \\ 2 & \cdot & 2 & \cdot \\ 1 & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \mu^o \\ \alpha_1^o \\ \alpha_2^o \\ \alpha_3^o \end{bmatrix} = \begin{bmatrix} 504 \\ 300 \\ 172 \\ 32 \end{bmatrix} .$$

With

$$\underset{\sim}{G} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad \text{a solution is} \quad \underset{\sim}{b}^o = \underset{\sim}{G} \begin{bmatrix} 504 \\ 300 \\ 172 \\ 32 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 88 \\ 32 \end{bmatrix} .$$

To convert this to a solution that corresponds to the restriction $\alpha_1 + \alpha_2 + \alpha_3 = 0$, we have $\underset{\sim}{P}' = [0 \;\; 1 \;\; 1 \;\; 1]$. Using this and

$$\mathbf{H} = \mathbf{GX'X} = \begin{bmatrix} 0 & . & . & . \\ 1 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & . & . & 1 \end{bmatrix}$$

of (4), we get from (9)

$$\mathbf{G}_r = \begin{bmatrix} 0 & . & . & . \\ . & \frac{1}{3} & . & . \\ . & . & \frac{1}{2} & . \\ . & . & . & 1 \end{bmatrix} - \begin{bmatrix} 1 & . & . & . \\ -1 & . & . & . \\ -1 & . & . & . \\ -1 & . & . & . \end{bmatrix} [-3 \quad 0 \quad 0 \quad 0]^{-} [0 \quad \tfrac{1}{3} \quad \tfrac{1}{2} \quad 1] \, .$$

Then since $[-3 \quad 0 \quad 0 \quad 0]^{-} = [-\frac{1}{3} \quad 0 \quad 0 \quad 0]'$, we have

$$\mathbf{G}_r = \begin{bmatrix} 0 & . & . & . \\ . & \frac{1}{3} & . & . \\ . & . & \frac{1}{2} & . \\ . & . & . & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} [0 \quad \tfrac{1}{3} \quad \tfrac{1}{2} \quad 1] = \frac{1}{18} \begin{bmatrix} 0 & 2 & 3 & 6 \\ 0 & 4 & -3 & -6 \\ 0 & -2 & 6 & -6 \\ 0 & -2 & -3 & 12 \end{bmatrix},$$

$$\mathbf{b}_r^o = \mathbf{G}_r\mathbf{X}'\mathbf{y} = \frac{1}{18} \begin{bmatrix} 0 & 2 & 3 & 6 \\ 0 & 4 & -3 & -6 \\ 0 & -2 & 6 & -6 \\ 0 & -2 & -3 & 12 \end{bmatrix} \begin{bmatrix} 504 \\ 300 \\ 172 \\ 32 \end{bmatrix} = \begin{bmatrix} 72\frac{2}{3} \\ 27\frac{1}{3} \\ 13\frac{1}{3} \\ -40\frac{2}{3} \end{bmatrix},$$

and

$$\mathbf{G}_r^{\#} = \mathbf{G}_r \begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & . & . \\ 2 & . & 2 & . \\ 1 & . & . & 1 \end{bmatrix} \mathbf{G}_r' = \frac{1}{54} \begin{bmatrix} 11 & -5 & -2 & 7 \\ -5 & 17 & -4 & -13 \\ -2 & -4 & 20 & -16 \\ 7 & -13 & -16 & 29 \end{bmatrix}.$$

It is easily seen that in $\mathbf{b}_r^o$

$$\alpha_1^o + \alpha_2^o + \alpha_3^o = 27\tfrac{1}{3} + 13\tfrac{1}{3} - 40\tfrac{2}{3} = 0 \, .$$

As a final comment it can be noted that usually $\underset{\sim}{P}'$ of the restrictions $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{0}$ in (2) has rank $p - r_{\underset{\sim}{X}}$; but it does not have to. Results (8) and (9) hold so long as $\underset{\sim}{P}'$ has full row rank, not exceeding $p - r_{\underset{\sim}{X}}$ and with its rows linearly independent of rows of $\underset{\sim}{X}$.

References

Mazumdar, Sati, Li, Ching Chun and Bryce, G. Rex. (1980). Correspondence between a linear restriction and a generalized inverse in linear model analysis. The American Statistician, 34, 103-105.

Searle, S. R. (1971). Linear Models. Wiley, New York.